

Optimal Investment with Transaction Costs under Cumulative Prospect Theory in Discrete Time

Bin Zou* and Rudi Zagst†
Chair of Mathematical Finance
Technical University of Munich

October 2015

Abstract

We study optimal investment problems with transaction costs under Kahneman and Tversky's cumulative prospective theory (CPT). A CPT investor makes investment decisions in a single-period discrete time financial market consisting of one risk-free asset and one risky asset, in which trading the risky asset incurs proportional costs. The objective is to seek the optimal investment to maximize the prospect value of the investor's final wealth. We have obtained explicit optimal investment to this problem in two examples. An economic analysis is conducted to investigate the impact of the transaction costs and risk aversion on the optimal investment.

Key Words: cumulative prospect theory; discrete time model; optimal investment; S-shaped utility; transaction costs; weighting function.

*Email: bin.zou@tum.de Phone: +49 89 289-17418

†Corresponding address: Chair of Mathematical Finance, Technical University of Munich, Parkring 11, Garching 85748, Germany. Email: zagst@tum.de Phone: +49 89 289-17401

1 Introduction

In economics and finance, an essential problem is how to model people's preference over uncertain outcomes. To address this problem, Bernoulli (1954) (originally published in 1738) proposed *expected utility theory* (EUT): any uncertain outcome X is represented by a numerical value $E[U(X)]$, which is the expected value of the utility $U(X)$ taken under an objective probability measure \mathbb{P} . If an individual prefers outcome X_1 to outcome X_2 , then $E[U(X_1)] \geq E[U(X_2)]$, and vice versa. Hence, under Bernoulli's setting (EUT), a rational individual seeks to maximize expected utility $E[U(X)]$ over all available choices. Bernoulli's original EUT was formally established by von Neumann and Morgenstern (thus the theory is also called *von Neumann-Morgenstern utility theorem*). Von Neumann and Morgenstern (1944) show that any individual whose behavior satisfies certain axioms has a utility function U and always prefers outcomes that maximize the expected utility. Since then, expected utility maximization has been one of the most widely used criteria for optimization problems concerning uncertainty, see, e.g., Merton (1969) and Samuelson (1969) for optimal investment problems (also called portfolio selection problems).

However, empirical experiment and research show that human behaviors may violate the basic tenets of EUT, e.g., Allais paradox challenges a fundamental axiom—*independence axiom*—of EUT. Please refer to Kahneman and Tversky (1979) for many designed choice problems whose results cannot be explained by EUT. Alternative theories are then proposed to address the drawbacks of EUT, such as *Prospect Theory* by Kahneman and Tversky (1979), *Rank-Dependent Utility* by Quiggin (1982), and *Cumulative Prospect Theory* (CPT) by Tversky and Kahneman (1992). CPT can explain diminishing sensitivity, loss aversion, and different risk attitudes. Furthermore, unlike prospect theory, CPT does not violate the first-order stochastic dominance. The detailed characterizations of CPT are presented in Subsection 2.2.

In this paper, we study optimal investment problems under the CPT framework. Jin and Zhou (2008) solved the problem in continuous time model (Black-Scholes model) by splitting into two Choquet optimization problems. The completeness of the market (the unique pricing kernel) plays an essential role in solving the problem in their paper, see also Carlier and Dana (2011). Optimal investment in single-period discrete model under CPT has been studied by Bernard and Ghossoub (2010), He and Zhou (2011), Pirvu

and Schulze (2012), and among others. Bernard and Ghossoub (2010) obtain explicit optimal solution in a frictionless financial market under the assumptions: piece-wise power utility, risk-free asset as the reference point and no short-selling constraint. They also study the properties of a new risk measure (called *CPT-ratio* in their paper) and conduct numerical simulations to investigate the impact of several factors, including mean, volatility, skewness, and risk aversion, on the optimal investment. He and Zhou (2011) consider the same problem as Bernard and Ghossoub (2010), but provide detailed analysis on the well-posedness of the problem by introducing a new measure of loss aversion (named *large-loss aversion degree* in their paper). They do not impose any constraint on investment strategy and are able to find optimal solution explicitly in two cases: (1) piece-wise power utility and risk-free asset as the reference point; (2) piece-wise linear utility and general reference point. Pirvu and Schulze (2012) generalize the previous work on this problem by considering a frictionless market consisting of one risk-free asset and multiple risky assets. Their main contribution is to provide a two-fund separation theorem between the risk-free asset and the market portfolio when the excess return has an elliptically symmetric distribution. Carassus and Rásonyi (2015) tackle the problem for the first time under a multiperiod market model. They not only address the well-posedness issue of the problem but also establish the existence of optimal strategies under some assumptions.

Without transaction costs, the optimal portfolio found in Merton's framework may lead to unrealistic strategies, e.g., buying stocks at infinite amount. In real life, transaction costs (bid-ask spread) are always present, albeit small for highly liquid assets. The seminal paper of Magill and Constantinides (1976) claim that optimal portfolio contains no-trade region through heuristic arguments. Davis and Norman (1990) provide rigorous treatment on optimal policies, solutions to the free boundary problem, and the value process (associated optimal expected utility). Shreve and Soner (1994) further generalize the results with viscosity techniques. Please refer to Kabanov and Safarian (2010) and references therein for a comprehensive introduction and development on mathematical theory of financial markets with transaction costs. The majority of existing literature on optimal investment problems with transaction costs, including those mentioned above, pursue analysis for an investor who behaves according to EUT.

In this paper, we consider optimal investment problems under the CPT framework in a discrete-time model with transaction costs, which, to our best

knowledge, has not been studied before. The main contribution of our work is to obtain explicit optimal investment strategy for a CPT investor under two examples.

The rest of the paper is organized as follows. Section 2 introduces the market model with transaction costs, and the three key components of the CPT framework. The main optimization problem is also formulated in Section 2. We review the utility functions and weighting functions used in the literature regarding CPT in Section 3. We then obtain optimal investment policies in explicit forms for two cases in Section 4 and Section 5, respectively. We provide economic analysis in Section 6 to study how optimal investment is affected by transaction costs and CPT preference. The conclusions of our work are summarized in Section 7.

2 The Setup

2.1 The Financial Market with Transaction Costs

We consider a single-period discrete-time financial market model equipped with a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In the model, time 0 and time T ($T > 0$) represent present and future, respectively. The financial market consists of one risk-free asset and one risky asset (e.g., stock). Trading the risk-free asset is frictionless. However, trading the risky asset will incur proportional transaction costs, and we denote such proportion by λ , where $\lambda \in (0, 1)$.

The return on the risk-free asset is r , where $r \geq 0$ is a constant. That means if an investor deposits 1\$ in the risk-free asset at time 0, he/she will receive $(1 + r)$ \$ at time T .

The (nominal) return on the risky asset is given by a random variable R . We assume the ask price of the risky asset $S(\cdot)$ is modeled by $S(T) = (1 + R) \cdot S(0)$, where $S(0)$ is a positive constant. The bid price of the risky asset at time t is given by $(1 - \lambda)S(t)$, where $t = 0, T$.

We assume \mathcal{F}_0 is trivial and \mathcal{F}_T is the completion of $\sigma(S(T))$. Thus R is \mathcal{F}_T measurable. For any \mathcal{F}_T measurable random variable Z , we denote its cumulative distribute function (CDF) by $F_Z(\cdot)$ and survival function by $S_Z(\cdot)$. By definition, $S_Z(\cdot) = 1 - F_Z(\cdot)$.

We consider an investor with initial portfolio (x_0, y_0) . That means the investor starts with x_0 and y_0 amount of money in the risk-free and the risky

asset, respectively. The investor chooses the amount of money to invest in the risky asset, denoted by θ , at time 0. The investor's position will be carried out to terminal time T and then liquidated. Denote the investor's terminal wealth after liquidation by $W(\theta)$. We calculate $W(\theta)$ based on the value of θ .

- $\theta = 0$

$$W(0) = \begin{cases} (1+r)x_0 + (1+R)y_0, & \text{if } y_0 < 0; \\ (1+r)x_0 + (1-\lambda)(1+R)y_0, & \text{if } y_0 \geq 0. \end{cases}$$

- $\theta > 0$ and $\theta + y_0 < 0$

$$W(\theta) = (1+r)x_0 + (1+R)y_0 + (R-r)\theta.$$

- $\theta > 0$ and $\theta + y_0 \geq 0$

$$W(\theta) = (1+r)x_0 + (1-\lambda)(1+R)y_0 + ((1-\lambda)(1+R) - (1+r))\theta.$$

- $\theta < 0$ and $\theta + y_0 < 0$

$$W(\theta) = (1+r)x_0 + (1+R)y_0 + (1+R - (1-\lambda)(1+r))\theta.$$

- $\theta < 0$ and $\theta + y_0 \geq 0$

$$W(\theta) = (1+r)x_0 + (1-\lambda)(1+R)y_0 + (1-\lambda)(R-r)\theta.$$

We can further write $W(\theta)$ in a universal form as

$$W(\theta) = (1+r)(x_0 - \theta) + (1+R)(y_0 + \theta) - \lambda \left[(1+R)(y_0 + \theta)^+ + (1+r)\theta^- \right],$$

where $x^+ := \max\{x, 0\}$ and $x^- := \max\{-x, 0\}$ for all $x \in \mathbb{R}$.

In this financial market, the non-arbitrage condition reads as

$$\mathbb{P}\left((1-\lambda)(1+R) < 1+r\right) > 0, \text{ and } \mathbb{P}\left(1+R > (1-\lambda)(1+r)\right) > 0. \quad (1)$$

Remark 2.1. We ignore two degenerated cases: (i) $(1-\lambda)(1+R) \equiv 1+r$, and (ii) $1+R \equiv (1-\lambda)(1+r)$, under which the market is also arbitrage-free. If $\lambda = 0$, meaning the market is frictionless, then the non-arbitrage condition (1) simply reduces to

$$0 < \mathbb{P}(R < r) < 1.$$

2.2 The CPT Framework

The cumulative prospect theory (CPT) provides a performance criterion for decision making under uncertainty. A CPT model is characterized by three key features.

- Reference point B

Experiments on people's behavior show that people make decisions based on comparison to some benchmark rather than final outcome. Hence, in CPT, a reference point is chosen to serve as the benchmark for evaluating a decision with uncertainty. Let X denote the final wealth of an investment decision. If $X > B$, $X - B$ are considered gains from the investment; while if $X < B$, $B - X$ are viewed as losses. For example, if B is set as 0, then the terminology of gain and loss fits into the common language.

- Utility functions $u_+(\cdot)$ and $u_-(\cdot)$

Investors are not universally risk averse, instead they have different risk attitudes towards gains and losses. In general, investors are more sensitive to potential losses than to potential gains. Such behavior is called *loss aversion* in economics and finance. To capture those features, CPT applies two different utility functions, u_+ and u_- , for gains and losses, respectively. Hence, the pathwise performance of an investment strategy (with associated wealth X) is defined by

$$u_+(X(\omega) - B) \cdot \mathbf{1}_{X(\omega) - B \geq 0} - u_-(B - X(\omega)) \cdot \mathbf{1}_{X(\omega) - B < 0}, \forall \omega \in \Omega,$$

where $\mathbf{1}_A$ is an indicator function of set A .

Tversky and Kahneman (1992) propose a S-shaped function v (called value function) to evaluate the gains and losses, where v is continuous and increasing throughout \mathbb{R} with $v(0) = 0$, but concave in \mathbb{R}^+ (for gains), and convex in \mathbb{R}^- (for losses). In our setting, the function v is given by

$$v(x) = u_+(x) \cdot \mathbf{1}_{x \geq 0} - u_-(x) \cdot \mathbf{1}_{x < 0}, \forall x \in \mathbb{R}.$$

Hence, we assume throughout this paper that the utility functions $u_{\pm} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are twice differentiable, strictly increasing, strictly concave and satisfy $u_{\pm}(0) = 0$. In addition, the loss aversion is represented by the following mathematical condition

$$u'_-(x) > u'_+(x) \text{ for all } x > 0. \quad (2)$$

- Weighting functions $w_+(\cdot)$ and $w_-(\cdot)$

Investors tend to overweight extreme events (small probability events) but underweight normal events (large probability events). This behavior is captured in CPT by transforming objective cumulative probabilities into subjective cumulative probabilities using weighting functions (also called distortion functions).

Denote the weighting functions of the probability of gains and losses by $w_+(\cdot)$ and $w_-(\cdot)$, respectively. We assume that $w_{\pm} : [0, 1] \rightarrow [0, 1]$ are strictly increasing and differentiable, and satisfy

$$w_{\pm}(0) = 0, \text{ and } w_{\pm}(1) = 1.$$

We define the CPT value of a random wealth X by

$$V(X) := \int_B^{\infty} u_+(x - B) d[-w_+(S_X(x))] - \int_{-\infty}^B u_-(B - x) d[w_-(F_X(x))].$$

Let $D := X - B$, then the above CPT value can be rewritten as

$$\begin{aligned} V(X) &= \int_0^{\infty} u_+(x) d[-w_+(S_D(x))] - \int_{-\infty}^0 u_-(-x) d[w_-(F_D(x))] \\ &:= V^+(X) - V^-(X). \end{aligned} \quad (3)$$

Through integration by parts and change of variable, we derive $V(X)$ in (3) as

$$V(X) = \int_0^{\infty} w_+(S_D(x)) du_+(x) - \int_0^{\infty} w_-(F_D(-x)) du_-(x). \quad (4)$$

2.3 The Problem

In the financial market described in Subsection 2.1, an investor selects investment strategy θ under the CPT framework introduced in Subsection 2.2. In other words, the investor wants to maximize the CPT value of his/her terminal wealth $V(W(\theta))$, which is defined by (3) or (4). Wealth $W(\theta)$ is a function of investment strategy θ , so is the CPT value $V(W(\theta))$. We then denote

$$J(\theta) := V(W(\theta)).$$

The reference point B can be random, and is given in the form of

$$B = a \cdot (1 + r) + b \cdot (1 + R), \text{ where } a, b \in \mathbb{R}. \quad (5)$$

B is a linear combination of the future values of \$1 investment in the risk-free asset and the risky asset. If $b = 0$, then the reference point B is a fixed constant.

In the above market setting, we implicitly assume that the investor we consider is a “small investor”, and his/her investment activities do not have an impact on the prices of assets. In addition, no investor in the real market has the capability to borrow the risk-free asset or short sell the risky asset at infinity amount. Those observations motivate us to constrain θ in a bounded region \mathcal{A} , namely,

$$\theta \in \mathcal{A} := [-\theta_m, \theta_M], \text{ where both } \theta_m, \theta_M \geq 0.$$

If the constrained optimal investment $\theta^* \in (-\theta_m, \theta_M)$, then θ^* is also optimal to the unconstrained problem. If $\theta^* = -\theta_m$ or θ_M , then the CPT criterion gives the investor “wrong” incentive to pursue maximal prospect by taking infinite risk if θ_m or $\theta_M \rightarrow \infty$. Hence, imposing the constraint $\theta \in \mathcal{A}$ helps the investor rule out the extreme decisions ($\theta = \pm\infty$), see Pirvu and Schulze (2012, Section 3.2).

To make sure $J(\theta)$ is well defined, we need the following assumption.

Assumption 2.1. *We assume both $V^+(W(\theta))$ and $V^-(W(\theta))$ are finite for all $\theta \in \mathcal{A}$.*

Proposition 2.1. *Assumption 2.1 is satisfied if one of the following conditions holds.*

- *The risky return R is bounded, e.g., R is a discrete random variable and $|R| \neq \infty$.*
- *The risky return R follows a normal distribution, or log-normal distribution, or student-t distribution, and for x small enough, there exists some $0 < \epsilon < 1$ such that*

$$w'_\pm(x) = O(x^{-\epsilon}), \text{ and } w'_\pm(1-x) = O(x^{-\epsilon}).$$

Proof. The first result is obvious. For the proof of the second result, please refer to He and Zhou (2011, Proposition 1) and Pirvu and Schulze (2012, Proposition 2.1). \square

We then formulate optimal investment problems with transaction costs under CPT as follows.

Problem 2.1. *In a financial market with transaction costs (as modeled in Subsection 2.1), an investor seeks optimal investment strategy to maximize the CPT value $J(\theta)$ of his/her terminal wealth. Equivalently, the investor seeks maximizer θ^* to the problem*

$$J(\theta^*) = \sup_{\theta \in \mathcal{A}} J(\theta) = \sup_{\theta \in \mathcal{A}} V(W(\theta)).$$

3 Review on Utility Function and Weighting Function

In a CPT model, excluding the reference point, there are two essential components: utility functions and weighting functions. We devote this section to the review of utility functions and weighting functions in the literature.

Tversky and Kahneman (1992) choose the following piece-wise power utility functions:

$$u_+(x) = x^\alpha, \text{ and } u_-(x) = kx^\beta, \quad (6)$$

where $k > 0$, $0 < \alpha, \beta \leq 1$. Power type utility function is the most commonly used one in optimal investment problems under CPT, see Barberis and Huang (2008), Bernard and Ghossoub (2010), He and Zhou (2011, Section 5.1), Pirvu and Schulze (2012, Section 4.1), and many others. The scaling property, $u_+(\theta x) = \theta^\alpha \cdot u_+(x)$, is the key to obtain explicit solution.

To make sure the utility functions (6) satisfy all the conditions proposed in Subsection 2.2, we need to impose certain assumptions on the parameters α , β and k .

Assumption 3.1. *(Power Utility Function) The utility functions are given by (6) with the parameters satisfying the following conditions:*

$$0 < \alpha \leq \beta < 1, \text{ and } k > 1.$$

Tversky and Kahneman (1992) estimate the values for the parameters as

$$\alpha = \beta = 0.88, \text{ and } k = 2.25,$$

which clearly satisfy the conditions imposed in Assumption 3.1.

Power utility function is unbounded, and hence may lead to an ill-posed problem (either infinite CPT value or infinite optimal investment), see He

and Zhou (2011) for detailed discussions. Another drawback of power utility function is that it fails to explain high risk averse behavior, as pointed in Rieger and Bui (2011). Hence, many researchers favor piece-wise exponential utility function as given in the assumption below, see Pirvu and Schulze (2012, Section 4.3).

Assumption 3.2. (*Exponential Utility Function*) *The utility functions are given by*

$$u_+(x) = 1 - e^{-\eta_+ x}, \text{ and } u_-(x) = \zeta (1 - e^{-\eta_- x}), \quad (7)$$

where $\eta_+, \eta_- > 0, \zeta > 1$.

He and Zhou (2011) claim that “the concavity/convexity condition (imposed on the utility functions) is insignificant, and hence can be ignored”. They then propose linear utility functions, see also Pirvu and Schulze (2012, Section 4.2),

$$u_+(x) = x, \text{ and } u_-(x) = kx.$$

However, if an investor’s preference is represented by linear utility functions, then the marginal utility is the same at any wealth level. According to linear utility preference, the pleasure of receiving 1 million is the same for a penniless investor and a billionaire, which clearly violates investor’s behavior as captured by *diminishing marginal utility*. Hence, we argue that it is inappropriate to consider linear utility under the CPT framework, at least from the economic point of view.

The weighting functions used in Tversky and Kahneman (1992) are given by

$$w_+(x) = \frac{x^\gamma}{(x^\gamma + (1-x)^\gamma)^{1/\gamma}}, \text{ and } w_-(x) = \frac{x^\delta}{(x^\delta + (1-x)^\delta)^{1/\delta}}. \quad (8)$$

As pointed in Rieger and Wang (2006), the above weighting functions may fail to be strictly increasing when $\gamma, \delta \leq 0.25$, but are indeed strictly increasing when $\gamma, \delta \geq 0.5$. The condition for strict increasing is relaxed to $\gamma, \delta \geq 0.28$ in Barberis and Huang (2008). The estimated values of parameters are $\gamma = 0.61$ and $\delta = 0.69$ in Tversky and Kahneman (1992), and thus satisfy the conditions for weighting functions introduced in Subsection 2.2.

Assumption 3.3. (*Tversky and Kahneman’s Weighting Function*) *The weighting functions are given by (8) with $\gamma, \delta \geq 0.28$.*

Prelec (1998) introduces the following weighting functions

$$w_+(x) = e^{-\delta^+(-\ln(x))^\gamma}, \text{ and } w_-(x) = e^{-\delta^-(-\ln(x))^\gamma}, \quad (9)$$

where $\gamma \in (0, 1)$ and $\delta^+, \delta^- > 0$. Rieger and Wang (2006) use Prelec's weighting function with $\delta^+ = \delta^- = 1$.

Assumption 3.4. (*Prelec's Weighting Function*) *The weighting functions are given by (9) with parameters $\delta_+ = \delta_- = 1$.*

Remark 3.1. *As pointed out in He and Zhou (2011) and Pirvu and Schulze (2012), if the weighting functions are as given in Assumption 3.3 or Assumption 3.4, then the second condition on weighting functions in Proposition 2.1 holds.*

4 Explicit Solution: First Example

To obtain explicit solution to Problem 2.1, we assume all the assumptions below hold in this section.

Assumption 4.1.

1. *The initial position on the risky asset is positive, $y_0 > 0$.*
2. *Short-selling is not allowed, $\theta_m \leq y_0$.*
3. *The reference point B is given by*

$$B = (1 + r)x_0 + (1 - \lambda)(1 + R)y_0.$$

We take $a = x_0$ and $b = (1 - \lambda)y_0$ in (5).

4. *The utility functions are power type as given in Assumption 3.1.*

Remark 4.1. *We make some comments on Assumption 4.1.*

- *Since $y_0 > 0$ and $\theta_m \leq y_0$, investors are allowed to sell the risky asset (θ can be negative), but no more than what they currently own. However, no short-selling constraint imposed in Bernard and Ghossoub (2010) is equivalent to $\theta \geq 0$.*

- We have $y_0 + \theta \geq 0$ for all $\theta \in \mathcal{A}$.
- The case of $y_0 < 0$ is less interesting since the no short-selling constraint then implies $\theta \geq 0$.
- For the given reference point B , we have $B = W(0)$. That means the benchmark we select is the terminal wealth of “doing nothing strategy”, which is an analogy to selecting the risk-free asset as the benchmark in a frictionless market.

First, due to $W(0) = B$, we obtain $J(0) = 0$.

Next, we study two sub-problems:

$$\sup_{0 \leq \theta \leq \theta_M} J(\theta) \dots\dots (P1), \text{ and } \sup_{-\theta_m \leq \theta \leq 0} J(\theta) \dots\dots (P2).$$

By comparing the optimal CPT values of the above two sub-problems, we obtain the solution to Problem 2.1.

4.1 Solution to Sub-Problem (P1)

If $\theta > 0$, then $y_0 + \theta > 0$, and hence the investor needs to sell all the holdings in the risky asset at liquidation.

Define random variable $Z_1 := (1 - \lambda)(1 + R) - (1 + r)$. Then simple calculation yields

$$D = W(\theta) - B = Z_1 \cdot \theta.$$

Define set A_1 by

$$A_1 := \{Z_1 < 0\} = \left\{1 + R < \frac{1 + r}{1 - \lambda}\right\}.$$

Notice that set A_1 is the set of losses for any long strategy, $\theta > 0$. Due to the non-arbitrage condition (1), $\mathbb{P}(A_1) > 0$.

By definition (3), the CPT value $J(\theta)$ in sub-problem (P1) is obtained by

$$J(\theta) = \int_0^\infty z^\alpha d[-w_+(S_{Z_1}(z))] \cdot \theta^\alpha - \int_{-\infty}^0 (-z)^\beta d[w_-(F_{Z_1}(z))] \cdot k\theta^\beta.$$

We define, for any \mathcal{F}_T measurable random variable Z , that

$$\begin{aligned} g_1(Z) &:= \int_0^\infty z^\alpha d[-w_+(S_Z(z))] = E[(Z^+)^\alpha \cdot w'_+(S_Z)], \\ l_1(Z) &:= \int_{-\infty}^0 (-z)^\beta d[w_-(F_Z(z))] = E[(Z^-)^\beta \cdot w'_-(F_Z)]. \end{aligned} \quad (10)$$

In general, $g_1(Z)$ (or $l_1(Z)$) can be understood as the prospect value of gains (or losses) of random wealth X with $X - B = Z$. In our setting here, $g_1(Z_1)$ and $l_1(Z_1)$ are exactly the prospect value of gains and the prospect value of losses (differ by a scalar k) of $W(1)$, which is the terminal wealth associated with the strategy $\theta = 1$. Mathematically, we have $g_1(Z_1) = V^+(W(1))$ and $k \cdot l_1(Z_1) = V^-(W(1))$, and hence both are finite due to Assumption 2.1.

With the definitions of g_1 and l_1 , the CPT value $J(\theta)$ is simplified as

$$J(\theta) = g_1(Z_1) \cdot \theta^\alpha - l_1(Z_1) \cdot k\theta^\beta, \quad \theta > 0.$$

Define $K_1(Z_1)$ by

$$K_1(Z_1) := \frac{g_1(Z_1)}{l_1(Z_1)}.$$

Given Assumption 2.1, $K_1(Z_1)$ is well defined and $K_1(Z_1) > 0$.

We summarize the solution to sub-problem (P1) below.

Theorem 4.1. *If Assumption 2.1 and Assumption 4.1 hold, then the optimal investment θ^* to sub-problem (P1) is obtained from one of the following scenarios.*

1. If $\mathbb{P}(A_1) = \mathbb{P}(Z_1 < 0) = 1$, then $\theta^* = 0$.

2. $0 < \mathbb{P}(A_1) < 1$

(a) If $\alpha = \beta$ and $k > \max\{1, K_1(Z_1)\}$, then $\theta^* = 0$.

(b) If $\alpha = \beta$ and $k = K_1(Z_1) > 1$, then $\theta^* = [0, \theta_M]$. That means any $\theta \in [0, \theta_M]$ is optimal.

(c) If $\alpha = \beta$ and $1 < k < K_1(Z_1)$, then $\theta^* = \theta_M$.

(d) If $\alpha < \beta$, then $\theta^* = \Theta_1 := \min\{\theta_1, \theta_M\}$, with θ_1 defined by

$$\theta_1 := \left(\frac{\alpha}{\beta k} K_1(Z_1) \right)^{\frac{1}{\beta - \alpha}}. \quad (11)$$

Proof. If $\mathbb{P}(A_1) = 1$, then the probability of suffering losses is 1 for all long strategies $\theta > 0$. Thus it is never optimal to buy the risky asset, $\theta^* = 0$. Mathematically, $\mathbb{P}(A_1) = \mathbb{P}(Z_1 < 0) = 1 \Rightarrow g_1(Z_1) = 0$. Then we have

$$J(\theta) = -l_1(Z_1) \cdot k\theta^\beta < J(0) = 0, \text{ for all } \theta > 0,$$

which directly indicates $\theta^* = 0$.

We next consider the non-trivial case: $0 < \mathbb{P}(A_1) < 1$.

Differentiating $J(\theta)$ gives

$$J'(\theta) = l_1(Z_1) \theta^{\alpha-1} [\alpha K_1(Z_1) - \beta k \theta^{\beta-\alpha}].$$

If $\alpha = \beta$, depending on the value of k , $J(\theta)$ is a strictly decreasing or increasing function or a constant, as summarized in (2a)-(2c). In Scenario (2c), $J(\theta)$ is strictly increasing, and $\lim_{\theta \rightarrow \infty} J(\theta) = +\infty$ (called ill-posed case in He and Zhou (2011)). So the constraint is binding, and we have $\theta^* = \theta_M$.

If $\alpha < \beta$, then θ_1 , defined by (11), is the unique solution to $J'(\theta) = 0$ on the positive axis. Furthermore, $J'(\theta) > 0$ for all $\theta \in (0, \theta_1)$ and $J'(\theta) < 0$ for all $\theta \in (\theta_1, \infty)$. Therefore, θ_1 is the unique maximizer to the problem $\sup_{\theta \geq 0} J(\theta)$. With the constraint $\theta \leq \theta_M$, the optimal investment $\theta^* = \min\{\theta_1, \theta_M\} := \Theta_1$. \square

4.2 Solution to Sub-Problem (P2)

Due to no short-selling constraint $\theta_m \leq y_0$, we have $y_0 + \theta \geq 0$ for all $-\theta_m \leq \theta \leq 0$. So the liquidation order at terminal T is to sell all the risky asset.

Define $Z_2 := (1 - \lambda)(R - r)$. We obtain in this case that

$$D = W(\theta) - B = Z_2 \cdot \theta.$$

Define set A_2 by $A_2 := \{Z_2 > 0\} = \{R > r\}$. Since investment θ is restricted to short strategies ($\theta \leq 0$) in this subsection, difference D is negative on set A_2 , meaning that set A_2 is the set of losses for any short strategy $\theta \in [-\theta_m, 0]$.

We define, for any \mathcal{F}_T measurable random variable Z , that

$$\begin{aligned} g_2(Z) &:= \int_{-\infty}^0 (-z)^\alpha d[w_+(F_Z(z))], \\ l_2(Z) &:= \int_0^\infty (z)^\beta d[-w_-(S_Z(z))]. \end{aligned} \tag{12}$$

The economic meanings of $g_2(Z)$ and $l_2(Z)$ are similar to those of $g_1(Z)$ and $l_1(Z)$, except the gains/losses are located on exactly opposite tails due to the different sign of θ .

The CPT value $J(\theta)$ then reads as

$$J(\theta) = g_2(Z_2) \cdot (-\theta)^\alpha - l_2(Z_2) \cdot k(-\theta)^\beta, \quad \text{for } -y_0 \leq \theta \leq 0,$$

which is well defined if Assumption 2.1 holds.

The unique solution to $J'(\theta) = 0$ on the whole negative axis is given by

$$\theta_2 := - \left(\frac{\alpha}{\beta k} K_2(Z_2) \right)^{\frac{1}{\beta-\alpha}}, \quad \text{where } K_2(Z_2) := \frac{g_2(Z_2)}{l_2(Z_2)}. \quad (13)$$

We directly provide the results to sub-problem (P2). Please refer to Theorem 4.1 for a similar proof.

Theorem 4.2. *If Assumption 2.1 and Assumption 4.1 hold, then the optimal investment θ^* to sub-problem (P2) is obtained from one of the following scenarios.*

1. If $\mathbb{P}(A_2) = 1$, then $\theta^* = 0$.
2. If $\mathbb{P}(A_2) = 0$, then $\theta^* = -\theta_m$.
3. $0 < \mathbb{P}(A_2) < 1$
 - (a) If $\alpha = \beta$ and $k > \max\{1, K_2(Z_2)\}$, then $\theta^* = 0$.
 - (b) If $\alpha = \beta$ and $k = K_2(Z_2) > 1$, then $\theta^* = [-\theta_m, 0]$.
 - (c) If $\alpha = \beta$ and $1 < k < K_2(Z_2)$, then $\theta^* = -\theta_m$.
 - (d) If $\alpha < \beta$, then $\theta^* = \Theta_2 := \max\{\theta_2, -\theta_m\}$.

4.3 Main Results

To find solution to Problem 2.1, we compare the optimal CPT value obtained from sub-problems (P1)-(P2) for different scenarios. Denote

$$K_M = \max\{K_1(Z_1), K_2(Z_2)\}.$$

The main results are summarized in the theorem below.

Theorem 4.3. *If Assumption 2.1 and Assumption 4.1 hold, then we have the following results for the optimal investment θ^* to Problem 2.1.*

1. If $\mathbb{P}(A_1) = 1$ or $\mathbb{P}(A_2) = 1$, then $\theta^* = 0$.
2. $0 < \mathbb{P}(A_1) < 1$ and $\mathbb{P}(A_2) = 0$
 - (a) If $\alpha = \beta$ and $k \geq \max\{1, K_1\}$, then $\theta^* = -\theta_m$.
 - (b) If $\alpha = \beta$ and $1 < k < K_1$, then $\theta^* = \arg \max_{\{-\theta_m, \theta_M\}} J(\theta)$.
 - (c) If $\alpha < \beta$, then $\theta^* = \arg \max_{\{-\theta_m, \Theta_1\}} J(\theta)$.
3. $0 < \mathbb{P}(A_1) < 1$ and $0 < \mathbb{P}(A_2) < 1$
 - (a) If $\alpha = \beta$ and $k > \max\{1, K_M\}$, then $\theta^* = 0$.
 - (b) If $\alpha = \beta$ and $k = K_1(Z_1) > \max\{1, K_2(Z_2)\}$, then $\theta^* = [0, \theta_M]$.
 - (c) If $\alpha = \beta$ and $k = K_2(Z_2) > \max\{1, K_1(Z_1)\}$, then $\theta^* = [-\theta_m, 0]$.
 - (d) If $\alpha = \beta$ and $k = K_1(Z_1) = K_2(Z_2) > 1$, then $\theta^* = [-\theta_m, \theta_M]$.
 - (e) If $\alpha = \beta$ and $1 < k < K_M$, then $\theta^* = \arg \max_{\{-\theta_m, \theta_M\}} J(\theta)$.
 - (f) If $\alpha < \beta$, then $\theta^* = \arg \max_{\{\Theta_1, \Theta_2\}} J(\theta)$.

Proof. Most results in the theorem are immediate consequences of Theorem 4.1 and Theorem 4.2.

In Scenario 2(a), we have

$$\sup_{0 \leq \theta \leq \theta_M} J(\theta) = 0, \text{ and } J(-\theta_m) = \sup_{-\theta_m \leq \theta \leq 0} J(\theta) > 0,$$

thus $\theta^* = -\theta_m$.

In Scenario 3(f), if the constraint is not binding, namely, $\Theta_1 = \theta_1$ and $\Theta_2 = \theta_2$, then we obtain finer results:

- (i) If $\alpha < \beta$ and $(g_1(Z_1))^\beta / (l_1(Z_1))^\alpha \geq (g_2(Z_2))^\beta / (l_2(Z_2))^\alpha$, then $\theta^* = \theta_1$.
- (ii) If $\alpha < \beta$ and $(g_1(Z_1))^\beta / (l_1(Z_1))^\alpha < (g_2(Z_2))^\beta / (l_2(Z_2))^\alpha$, then $\theta^* = \theta_2$.

The above results are based on comparing $J(\theta_1)$ with $J(\theta_2)$, see He and Zhou (2011, Appendix). \square

Using the CPT definition (4), we rewrite $g_2(Z_2)$ and $l_2(Z_2)$ as

$$g_2(Z_2) = \int_0^\infty w_+(F_{Z_2}(-z))du_+(z), \quad l_2(Z_2) = \frac{1}{k} \int_0^\infty w_-(S_{Z_2}(z))du_-(z),$$

and $g_1(Z_1)$ and $l_1(Z_1)$ as

$$g_1(Z_1) = \int_0^\infty w_+(S_{Z_1}(z))du_+(z), \quad l_1(Z_1) = \frac{1}{k} \int_0^\infty w_-(F_{Z_1}(-z))du_-(z).$$

We have $Z_2 > Z_1$ almost surely, and then $F_{Z_2}(z) \leq F_{Z_1}(z)$ for all z (strict inequality holds for some z). Furthermore, if Z_2 is symmetrically distributed around 0 (equivalently, R is symmetrically distributed around r), we have

$$\begin{aligned} F_{Z_2}(-z) &= 1 - F_{Z_2}(z) \geq 1 - F_{Z_1}(z) = S_{Z_1}(z), \\ F_{Z_1}(-z) &= 1 - S_{Z_1}(-z) \geq 1 - F_{Z_2}(z) = S_{Z_2}(z). \end{aligned}$$

Therefore $g_2(Z_2) > g_1(Z_1)$ and $l_2(Z_2) < l_1(Z_1)$. Consequently, $K_2(Z_2) > K_1(Z_1)$ holds, and then $K_M = K_2(Z_2)$.

Comparing Theorem 4.3 with the results in a frictionless market (see, for instance, Bernard and Ghossoub (2010, Theorem 3.1) and He and Zhou (2011, Theorem 3)), there are several differences:

- if $\lambda = 0$, then $0 < \mathbb{P}(A_1) < 1$ and $0 < \mathbb{P}(A_2) < 1$, so Cases (1)-(2) in Theorem 4.3 will never happen.
- if $\lambda = 0$, then $Z_1 = Z_2 = R - r$. However, with $\lambda > 0$, we have $Z_1 < Z_2 < R - r$. If Z_2 is symmetrically distributed around 0, we have $K_1(Z_1) = K_2(Z_2)$ if $\lambda = 0$, but $K_1(Z_1) < K_2(Z_2)$ if $\lambda > 0$.

4.4 Discussions for $y_0 = 0$

If the investor does not hold any risky asset at time 0 ($y_0 = 0$), we can remove the constraint of no short-selling and still obtain explicit solution. Notice that $B = (1 + r)x_0$ when $y_0 = 0$, which is the most common choice for the reference point and is used by Bernard and Ghossoub (2010), He and Zhou (2011), Pirvu and Schulze (2012), and many others.

The solution to sub-problem (P1) is exactly the same as in Theorem 4.3. However, the solution to sub-problem (P2) here is different from the results

in Theorem 4.2. Since $y_0 = 0$, we have $y_0 + \theta \leq 0$ for all $\theta \in [-\theta_m, 0]$ (but $y_0 + \theta \geq 0$ in Subsection 4.2).

Define $Z_3 := 1 + R - (1 - \lambda)(1 + r)$. Then we have

$$D = W(\theta) - B = Z_3 \cdot \theta \text{ for all } \theta \in [-\theta_m, 0].$$

Define the set of losses A_3 by

$$A_3 := \{Z_3 > 0\} = \{1 + R > (1 - \lambda)(1 + r)\}.$$

The non-arbitrage condition (1) implies $\mathbb{P}(A_3) > 0$, while $\mathbb{P}(A_2) = 0$ is possible in Subsection 4.2.

By replacing Z_2 by Z_3 , A_2 by A_3 and removing the scenario of $\mathbb{P}(A_2) = 0$ in Theorem 4.3, we obtain the optimal investment to Problem 2.1 in the case of $y_0 = 0$.

To study the connection between those two cases, we modify the notations for $J(\theta)$. For initial position (x_0, y_0) with $y_0 > 0$, we denote $J(\theta, x_0, y_0) = J(\theta)$. For initial position x_0 and $y_0 = 0$, we denote $\tilde{J}(\theta, x_0) = J(\theta)$. If an investor has initial wealth of amount $x_0 + y_0$, then he/she can buy the risky asset of amount y_0 and hold portfolio (x_0, y_0) . On the other hand, if an investor holds portfolio (x_0, y_0) at the beginning, he/she can liquidate the risky asset and deposit all the money, $x_0 + (1 - \lambda)y_0$, in the risk-free asset. Therefore, we obtain

$$\tilde{J}(\tilde{\theta}_1^*, x_0 + (1 - \lambda)y_0) \leq J(\theta^*, x_0, y_0) \leq \tilde{J}(\tilde{\theta}_2^*, x_0 + y_0),$$

where $\tilde{\theta}_1^*$, θ^* , and $\tilde{\theta}_2^*$ are the optimal investment to the corresponding initial position.

5 Explicit Solution: Second Example

In this section, we consider a binomial market specified by

$$1 + R = \begin{cases} u, & \text{with probability } 1 - p \\ d, & \text{with probability } p \end{cases}, \quad (14)$$

where $u > d > 0$ and $0 < p < 1$.

The non-arbitrage condition (1) in this model reads as

$$u > (1 - \lambda)(1 + r) > (1 - \lambda)^2 d.$$

Given a payoff $\xi \in \mathcal{F}_T$, we have

$$\xi = \begin{cases} \xi_u, & \text{when } 1 + R = u; \\ \xi_d, & \text{when } 1 + R = d. \end{cases} \quad (15)$$

In what follows, we may denote $\xi = (\xi_u, \xi_d)$ in the above sense. In the market modeled by (14), assume we can replicate ξ by strategy θ_ξ and initial amount x_ξ .

- If $\xi_u \geq \xi_d$, then we obtain

$$\theta_\xi = \frac{\xi_u - \xi_d}{(1 - \lambda)(u - d)}, \quad (16)$$

$$x_\xi = \frac{1}{1 + r}(p_u^b \cdot \xi_u + p_d^b \cdot \xi_d), \quad (17)$$

where p_u^b and p_d^b are defined by

$$p_u^b := \frac{(1 + r) - (1 - \lambda)d}{(1 - \lambda)(u - d)}, \quad p_d^b := \frac{(1 - \lambda)u - (1 + r)}{(1 - \lambda)(u - d)}. \quad (18)$$

- If $\xi_u < \xi_d$, then we obtain

$$\theta_\xi = \frac{\xi_u - \xi_d}{(u - d)}, \quad (19)$$

$$x_\xi = \frac{1}{(1 + r)(1 - \lambda)}(p_u^s \cdot \xi_u + p_d^s \cdot \xi_d), \quad (20)$$

where p_u^s and p_d^s are defined by

$$p_u^s := \frac{(1 - \lambda)(1 + r) - d}{u - d}, \quad p_d^s := \frac{u - (1 - \lambda)(1 + r)}{u - d}. \quad (21)$$

Remark 5.1. If $\xi_u \geq \xi_d$ (or $\xi_u < \xi_d$), the replication strategy involves buying (or selling) the risky asset (since $\theta_\xi \geq 0$ in (16) and $\theta_\xi < 0$ in (19)).

Notice that $p_u^b + p_d^b = 1$, $p_u^s + p_d^s = 1$ and $p_u^b, p_d^b > 0$, but p_d^b and p_u^s may be negative, so (p_u^b, p_d^b) and (p_u^s, p_d^s) are not necessary risk neutral probability measures. However, if $\lambda = 0$, we have $p_u^b = p_u^s$, $p_d^b = p_d^s$, and (p_u^b, p_d^b) indeed the unique risk neutral probability measure.

Since we impose trading constraint $\theta \in \mathcal{A} = [-\theta_m, \theta_M]$, the replication strategy θ_ξ , given by (16) or (19), may not be attainable under the constraint. In this section, we shall study Problem 2.1 without constraint and let the unconstrained solution suggest whether the constraint is binding.

To solve Problem 2.1, we assume the assumptions below hold throughout this section.

Assumption 5.1.

1. *The investor begins with initial portfolio $(x_0, 0)$. The investor does not hold any risky asset at the beginning, $y_0 = 0$.*
2. *The risky return in the market is modeled by (14).*
3. *The reference point is given by*

$$B = (1 + r)x_0.$$

4. *The utility functions are exponential type as given in Assumption 3.2 with $\eta_+ = \eta_- = \eta$.*

Due to the analysis above on the replication of a random payoff ξ , we consider two sets of random payoffs:

$$\begin{aligned}\Xi^b &:= \{\xi = (\xi_u, \xi_d) \in \mathcal{F}_T : \xi_u \geq \xi_d, p_u^b \cdot (\xi_u - B) + p_d^b \cdot (\xi_d - B) = 0\}, \\ \Xi^s &:= \{\xi = (\xi_u, \xi_d) \in \mathcal{F}_T : \xi_u < \xi_d, p_u^s \cdot (\xi_u - B) + p_d^s \cdot (\xi_d - B) = 0\}.\end{aligned}$$

To solve the maximization problem $\sup_{\theta \in \mathbb{R}} J(\theta)$, we consider two sub problems:

$$\sup_{\xi \in \Xi^b} V(\xi) \dots \dots \dots (\text{P3}) \text{ and } \sup_{\xi \in \Xi^s} V(\xi) \dots \dots \dots (\text{P4}).$$

5.1 Solution to Sub-Problem (P3)

If $p_d^b \leq 0$ (corresponding to $\mathbb{P}(A_1) = 1$ in Subsection 4.1), then $V(\xi) \leq 0 = V((B, B))$. Hence $\xi^* = (B, B) \in \Xi^b$, and $\theta^* = 0$ because of (16).

If $p_d^b \in (0, 1)$, we immediately have $\xi_d - B \leq 0 \leq \xi_u - B$. $\forall \xi \in \Xi^b$, $B - \xi_d = \frac{p_u^b}{p_d^b}(\xi_u - B)$. By the definition of CPT, we write $V(\xi)$ as

$$\begin{aligned}V(\xi) &= w_+(1 - p) \cdot u_+(\xi_u - B) - w_-(p) \cdot u_-(B - \xi_d) \\ &= w_+(1 - p) \cdot u_+(\xi_u - B) - w_-(p) \cdot u_-\left(\frac{p_u^b}{p_d^b}(\xi_u - B)\right) := L^b(\xi_u).\end{aligned}$$

Then sub-problem (P3) is equivalent to $\sup_{\xi_u \geq B} L^b(\xi_u)$.

- $p_u^b = p_d^b$

In this case, we rewrite $L^b(\xi_u)$ as

$$L^b(\xi_u) = (w_+(1-p) - \zeta w_-(p)) u_+(\xi_u - B).$$

Define a threshold $\bar{\zeta}$ by

$$\bar{\zeta} := \frac{p_d^b \cdot w_+(1-p)}{p_u^b \cdot w_-(p)}.$$

Notice $\bar{\zeta} = w_+(1-p)/w_-(p)$ when $p_u^b = p_d^b$.

Therefore, we obtain the optimal payoff ξ_u^* by

$$\xi_u^* = \begin{cases} B, & \text{when } \zeta > \bar{\zeta} \\ [B, +\infty), & \text{when } \zeta = \bar{\zeta} > 1. \\ +\infty, & \text{when } 1 < \zeta < \bar{\zeta} \end{cases}$$

Hence, using (16) and $\xi_d = 2B - \xi_u$, the optimal investment θ^* in $[0, \theta_M]$ is given by

$$\theta^* = \begin{cases} 0, & \text{when } \zeta > \bar{\zeta} \\ [0, \theta_M], & \text{when } \zeta = \bar{\zeta} > 1. \\ \theta_M, & \text{when } 1 < \zeta < \bar{\zeta} \end{cases} \quad (22)$$

- $p_u^b > p_d^b$

We calculate $(L^b)'(\xi_u)$ as

$$(L^b)'(\xi_u) = w_+(1-p) \cdot \eta e^{-\eta(\xi_u-B)} \left(1 - \frac{\zeta}{\bar{\zeta}} e^{-\eta(p_u^b/p_d^b-1)(\xi_u-B)} \right).$$

If $\zeta \leq \bar{\zeta}$, then $(L^b)'(\xi_u) > 0$ for all $\xi_u > B$. The strict increasing property of prospect with respect to ξ_u (and thus θ) yields the optimal investment in $[0, \theta_M]$ is $\theta^* = \theta_M$.

If $\zeta > \bar{\zeta}$, then we have

$$\lim_{\xi_u \rightarrow B} (L^b)'(\xi_u) < 0, \text{ and } \lim_{\xi_u \rightarrow +\infty} (L^b)'(\xi_u) = 0.$$

Furthermore, $(L^b)'(\xi_u)$ changes sign only once in the interval $[B, +\infty)$. The constraint $\theta \in [0, \theta_M]$ is equivalent to $\xi_u \in [B, \xi_M]$, where ξ_M is defined through

$$\frac{1}{(1-\lambda)(u-d)} \left[\left(\frac{p_u^b}{p_d^b} + 1 \right) (\xi_M - B) \right] = \theta_M.$$

Hence, we have

$$\sup_{\theta \in [0, \theta_M]} J(\theta) = \max\{L^b(B) = 0, L^b(\xi_M)\}.$$

To summarize, if $p_u^b > p_d^b$, the optimal payoff ξ_u^* in $[B, \xi_M]$ is given by

$$\xi_u^* = \arg \max_{\{B, \xi_M\}} \{L^b(B), L^b(\xi_M)\},$$

and the optimal investment θ^* in $[0, \theta_M]$ is obtained as

$$\theta^* = \begin{cases} 0, & \text{if } \xi_u^* = B \\ \theta_M, & \text{if } \xi_u^* = \xi_M \end{cases} \quad (23)$$

or in the expression of

$$\theta^* = \arg \max_{\{0, \theta_M\}} J(\theta).$$

- $p_u^b < p_d^b$

Due to the analysis above, we easily obtain that $(L^b)'(\xi_u) < 0$ when $\zeta \geq \bar{\zeta}$, and thus $\theta^* = 0$ in this scenario.

If $\zeta < \bar{\zeta}$, solving $(L^b)'(\xi_u) = 0$ gives a unique maximizer

$$\xi_u^* = B + \frac{p_d^b}{\eta(p_d^b - p_u^b)} \ln \left(\frac{\bar{\zeta}}{\zeta} \right),$$

and then

$$\begin{aligned} \theta_3 &:= \frac{1}{(1-\lambda)(u-d)} \left[\left(\frac{p_u^b}{p_d^b} + 1 \right) (\xi_u^* - B) \right] \\ &= \frac{1}{\eta((1-\lambda)(u+d) - 2(1+r))} \ln \left(\frac{\bar{\zeta}}{\zeta} \right). \end{aligned} \quad (24)$$

Notice that $\theta_3 > 0$ since $p_u^b < p_d^b$ and $\zeta < \bar{\zeta}$.

Therefore, the optimal investment θ^* in $[0, \theta_M]$ can be summarized as

$$\theta^* = \begin{cases} 0, & \text{if } \zeta \geq \bar{\zeta} \\ \Theta_3 := \min\{\theta_3, \theta_M\}, & \text{if } \zeta < \bar{\zeta} \end{cases}. \quad (25)$$

5.2 Solution to Sub-Problem (P4)

If $p_u^s \leq 0$, then $V(\xi) \leq 0$ for all $\xi \in \Xi^s$, hence $\theta^* = 0$.

If $p_u^s > 0$, then $\forall \xi \in \Xi^s$, we have $\xi_u - B \leq 0 \leq \xi_d - B$, and

$$\begin{aligned} J(\theta) &= V(\xi) = w_+(p)u_+(\xi_d - B) - w_-(1-p)u_-(B - \xi_u) \\ &= w_+(p)u_+(\xi_d - B) - w_-(1-p)u_- \left(\frac{p_d^s}{p_u^s}(\xi_d - B) \right) := L^s(\xi_d). \end{aligned}$$

The first derivative of $L^s(\xi_d)$ is calculated as

$$(L^s)'(\xi_d) = \eta e^{-\eta(\xi_d - B)} w_+(p) \left[1 - \frac{\zeta}{\underline{\zeta}} e^{-\eta \left(\frac{p_d^s}{p_u^s} - 1 \right) (\xi_d - B)} \right],$$

where constant $\underline{\zeta}$ is defined by

$$\underline{\zeta} := \frac{p_u^s \cdot w_+(p)}{p_d^s \cdot w_-(1-p)}.$$

By (19) and the budget constraint (20), we derive

$$\theta = -\frac{(p_d^s/p_u^s + 1)(\xi_d - B)}{u - d},$$

which implies that

$$\text{sign}(J'(\theta)) = \text{sign} \left((L^s)'(\xi_d) \cdot \frac{d\xi_d}{d\theta} \right) = -\text{sign}((L^s)'(\xi_d))$$

It is obvious that sub-problem (P4) and $\sup_{\xi_d \geq B} L^s(\xi_d)$ are equivalent. The analysis is the same as for $\sup_{\xi_u \geq B} L^b(\xi_u)$ in the previous subsection, and we summarize results below.

- $p_u^s = p_d^s$

In this case, we have

$$\text{sign}((L^s)'(\xi_d)) = \text{sign}\left(1 - \frac{\zeta}{\underline{\zeta}}\right) = -\text{sign}(J'(\theta)).$$

$$\theta^* = \begin{cases} 0, & \text{when } \zeta > \underline{\zeta} \\ [-\theta_m, 0], & \text{when } \zeta = \underline{\zeta} > 1. \\ -\theta_m, & \text{when } 1 < \zeta < \underline{\zeta} \end{cases} \quad (26)$$

- $p_u^s < p_d^s$

If $\zeta \leq \underline{\zeta}$, then $(L^s)'(\xi_d) > 0$ for all $\xi_d > B$ and $J'(\theta) < 0$ for all $\theta < 0$. Hence $\theta^* = -\theta_m$.

If $\zeta > \underline{\zeta}$, then

$$(L^s)'(B) < 0, \text{ and } \lim_{\theta \rightarrow +\infty} (L^s)'(\xi_d) = 0.$$

In addition, $(L^s)'(\xi_d)$ changes sign once in $[B, \infty)$. Therefore,

$$\sup_{\theta \in [-\theta_m, 0]} J(\theta) = \max\{J(0) = 0, J(-\theta_m)\},$$

and the optimal investment θ^* is given by

$$\theta^* = \arg \max_{\{-\theta_m, 0\}} J(\theta). \quad (27)$$

- $p_u^s > p_d^s$

If $\zeta \geq \underline{\zeta}$, then $(L^s)'(\xi_d) < 0$ for all $\xi_d > B$, so $\theta^* = 0$.

If $\zeta < \underline{\zeta}$, then $(L^s)'(B) > 0$, and there exists a unique maximizer ξ_d^* in $[B, \infty)$ which solves $(L^s)'(\xi_d) = 0$

$$\xi_d^* = B + \frac{p_u^s}{\eta(p_u^s - p_d^s)} \ln\left(\frac{\underline{\zeta}}{\zeta}\right).$$

For (ξ_u^*, ξ_d^*) , where ξ_u^* satisfies $p_u^s \cdot (\xi_u^* - B) + p_d^s \cdot (\xi_d^* - B) = 0$, the corresponding replication strategy is given by

$$\theta_4 = -\frac{1}{\eta(2(1-\lambda)(1+r) - (u+d))} \ln\left(\frac{\underline{\zeta}}{\zeta}\right). \quad (28)$$

We then obtain the optimal investment in $[-\theta_m, 0]$

$$\theta^* = \begin{cases} 0, & \text{if } \zeta \geq \underline{\zeta} \\ \Theta_4 := \max\{\theta_4, -\theta_m\}, & \text{if } \zeta < \underline{\zeta} \end{cases}. \quad (29)$$

5.3 Main Results

To obtain explicit solution to Problem 2.1, it remains to compare the optimal prospect values obtained in the previous two subsections case by case.

A major difference between power utility (Assumption 3.1) and exponential utility (Assumption 3.2) is that the CPT value under exponential utility is always finite, due to the fact that $0 \leq u_{\pm}(x) \leq \zeta$ for all $x \geq 0$. Hence Assumption 2.1 is always satisfied under exponential utility.

Theorem 5.1. *Let Assumption 5.1 hold, we then obtain the optimal investment θ^* to Problem 2.1 through the following results.*

1. If $p_d^b \leq 0$, then $\sup_{\theta \in [-\theta_m, \theta_M]} J(\theta) = \sup_{\theta \in [-\theta_m, 0]} J(\theta)$, and θ^* is given by (26), (27), or (29) depending on the comparison of p_u^s and p_d^s .
2. If $p_u^s \leq 0$, then $\sup_{\theta \in [-\theta_m, \theta_M]} J(\theta) = \sup_{\theta \in [0, \theta_M]} J(\theta)$, and θ^* is given by (22), (23), or (25) depending on the comparison of p_u^b and p_d^b .
3. If $p_d^b > 0$ and $p_u^s > 0$, or equivalently $d < (1 - \lambda)(1 + r) < \frac{1+r}{1-\lambda} < u$, we further separate the discussions as follows.

$$(a) \ p_u^b = p_d^b \ (\Rightarrow p_u^s < p_d^s)$$

$$J(\theta^*) = \begin{cases} 0, & \text{if } \zeta \geq \bar{\zeta}, \zeta > \underline{\zeta}, J(-\theta_m) < 0; \\ \max\{J(-\theta_m), J(\theta_M)\}, & \text{otherwise.} \end{cases}$$

$$(b) \ p_u^s = p_d^s \ (\Rightarrow p_u^b > p_d^b)$$

$$J(\theta^*) = \begin{cases} 0, & \text{if } \zeta > \bar{\zeta}, \zeta \geq \underline{\zeta}, J(\theta_M) < 0; \\ \max\{J(-\theta_m), J(\theta_M)\}, & \text{otherwise.} \end{cases}$$

$$(c) \ p_u^b > p_d^b \text{ and } p_u^s < p_d^s$$

$$J(\theta^*) = \max\{J(-\theta_m), J(0), J(\theta_M)\}.$$

(d) $p_u^b > p_d^b$ and $p_u^s > p_d^s$

$$J(\theta^*) = \max\{J(\Theta_4), J(0), J(\theta_M)\}.$$

(e) $p_u^b < p_d^b$ and $p_u^s < p_d^s$

$$J(\theta^*) = \max\{J(-\theta_m), J(0), J(\Theta_3)\}.$$

If $\lambda \geq \bar{\lambda} := \max\{1 - \frac{1+r}{u}, 1 - \frac{d}{1+r}\}$, both $p_d^b, p_u^s \leq 0$, then by Theorem 5.1, the optimal investment $\theta^* = 0$. This result shows the optimal investment largely depends on transaction costs. CPT investors will not trade the risky asset as long as λ is above the threshold $\bar{\lambda}$.¹ However, if there are no transaction costs in the market ($\lambda = 0$), then the non-arbitrage condition $d < 1 + r < u$ implies that $\bar{\lambda} > \lambda = 0$.

6 Economic Analysis

In this section, we conduct economic analysis to study how optimal investment is affected by various factors, such as transaction costs and risk aversion. The calculations in Section 5 are straightforward as long as the binomial model (14) has been estimated. However, under Tversky and Kahneman's weighting functions (8) or Prelect's weighting functions (9), the numerical calculations for $K_1(Z_1)$ and $K_2(Z_2)$ (two integrals) in Section 4 are very complicate even the risky return $1 + R$ is normally distributed or lognomarilly distributed. In what follows, we obtain numerical results based on the model in Section 4 and conclusions from Theorem 4.3.

6.1 Data and Model Parameters

We consider optimal investment problems in a single-period discrete model, hence we should select a relatively short time window. In the economic analysis thereafter, we select time window to be 1 week, $T = 1$ week.

To estimate the risk-free interest rate r , we use the data of 3-month U.S. Treasury Bill traded in the secondary market between January 6, 2012 and August 28, 2015. There are 191 observations during the selected time period. The mean and the median of the annual return are 0.0541% and 0.05%. The

¹ $\bar{\lambda}$ is calculated to be 1.25% using the weekly data example in Subsection 6.1.

right skewness of the data suggests us to choose median as the annual return of the risk-free asset. Then the weekly risk-free return r is estimated by

$$r = (1 + 0.05\%)^{1/52} - 1 = 9.6130 \times 10^{-6}.$$

In order to estimate the distribution of the risky return R , we choose the weekly adjusted price of S&P 500 from January 3, 2012 to August 31, 2015. Based on the 191 observations, we estimate the mean and the standard deviation of R as

$$\mu = 0.002398, \text{ and } \sigma = 0.01573.$$

If we consider log return, $\ln(1 + R)$, the estimations for the mean and the standard deviation are

$$\mu = 0.002272, \text{ and } \sigma = 0.01574.$$

Both $1 + R$ and $\ln(1 + R)$ are approximately normal according to their QQ plots. Two QQ plots are very similar, and we provide the QQ plot of $1 + R$ in Figure 1.

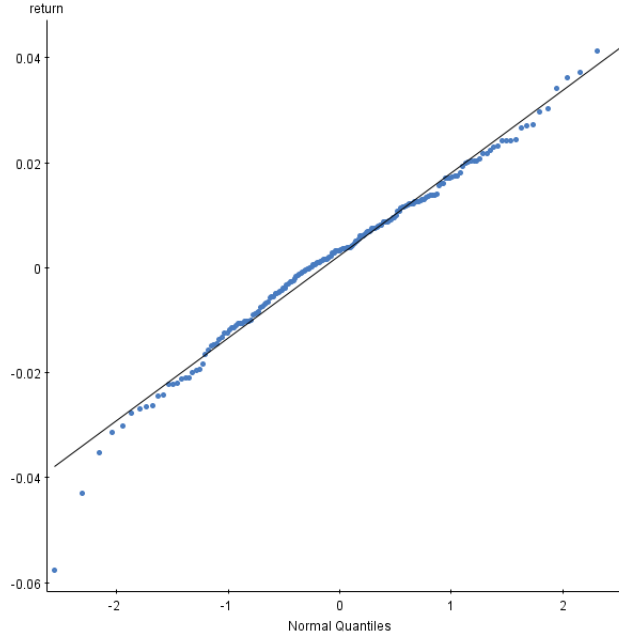


Figure 1: QQ Plot of the Risky Return R (Weekly Data)

For the numerical calculations in this section, we select Tversky and Kahneman's weighting function, specified as in Assumption 3.3, with parameters

$$\gamma = 0.61, \text{ and } \delta = 0.69.$$

The utility functions are of power utility type, namely, $u_+(x) = x^\alpha$, and $u_-(x) = kx^\beta$. The risk attitudes of an CPT investor depend on α and β . We separate the discussions into two cases: $\alpha = \beta$ and $\alpha < \beta$.

6.2 The Case of $\alpha = \beta$

If $\alpha = \beta$, the optimal investment is of one of the trivial solutions, see (2a), (2b), and (3a)-(3e) in Theorem 4.3. In the analysis, we select $\alpha = \beta = 0.88$, as estimated in Tversky and Kahneman (1992).

If $1+R$ (or $\ln(1+R)$) is normally distributed, we have $0 < \mathbb{P}(A_1), \mathbb{P}(A_2) < 1$. Then according to Case (3) in Theorem 4.3, we need to calculate $K_1(Z_1)$ and $K_2(Z_2)$ in order to obtain the optimal investment θ^* . Calculations are done when $1+R$ follows a normal distribution, and the graphs of $K_1(Z_1)$ and $K_2(Z_2)$ as a function of transaction costs proportion λ are provided in Figure 2 and Figure 3, respectively. The graphs of $K_1(Z_1)$ and $K_2(Z_2)$ are very similar when $\ln(1+R)$ is normally distributed.

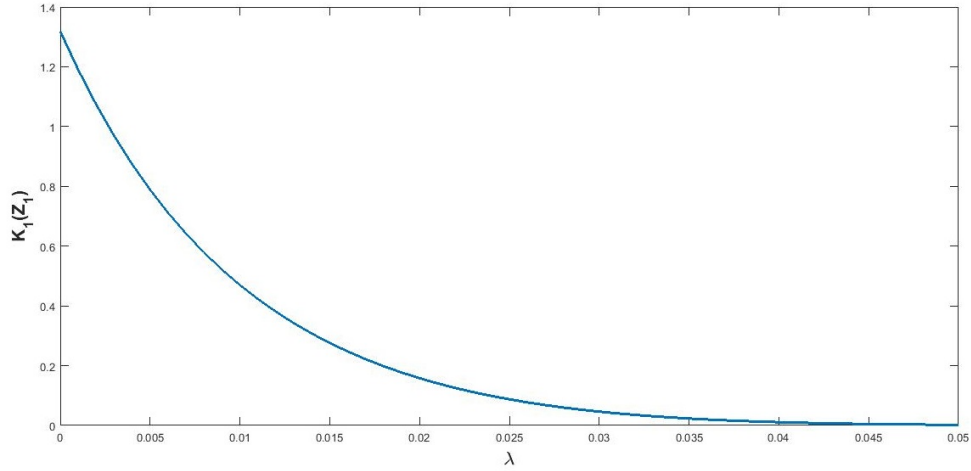


Figure 2: $K_1(Z_1)$ when $0 < \lambda < 5\%$

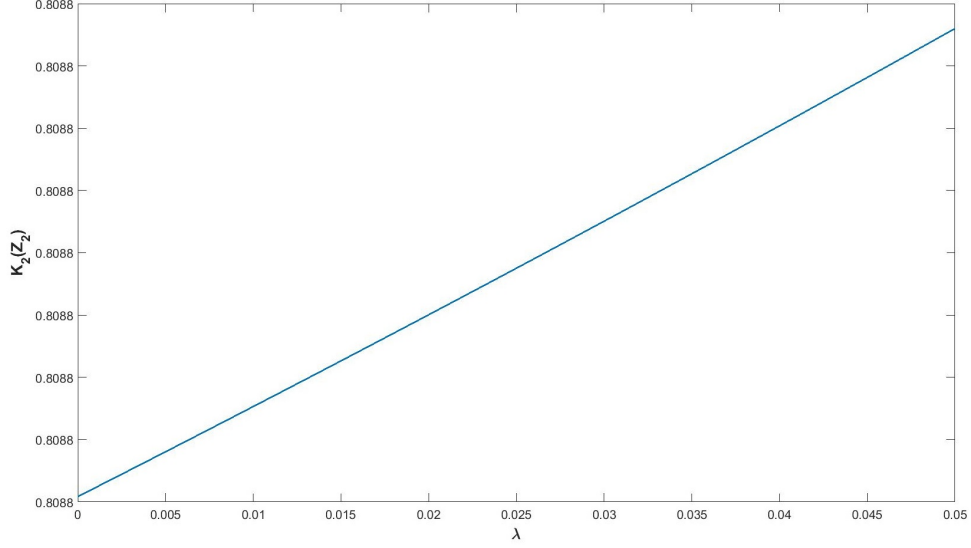


Figure 3: $K_2(Z_2)$ when $0 < \lambda < 5\%$

If λ increases ($\lambda \uparrow$), both Z_1 and Z_2 will decrease ($Z_1 \downarrow$ and $Z_2 \downarrow$). Then immediately, we obtain $F_{Z_1} \uparrow$, $S_{Z_1} \downarrow$, $F_{Z_2} \uparrow$, and $S_{Z_2} \downarrow$. All these results together imply that $K_1(Z_1) \uparrow$ and $K_2(Z_2) \downarrow$, as confirmed by Figure 2 and Figure 3.

From Figures 2 and 3, we observe more interesting results. First, the magnitude of change with respect to λ is significantly different for $K_1(Z_1)$ and $K_2(Z_2)$. As λ increases from 0 to 5%, $K_1(Z_1)$ drops from around 1.3 to nearly 0 but $K_2(Z_2)$ increases less than 10^{-5} (Please notice that the first 4 decimal places are exactly the same along the y-axis in Figure 3). In addition, $K_1(Z_1) > 1$ when λ is small enough; however, $K_2(Z_2) < 1$ for all $\lambda \in (0, 5\%)$. In Tversky and Kahneman (1992), k is estimated to be 2.25, then we have Scenario 3(a) in Theorem 4.3, and hence the optimal investment $\theta^* = 0$.

In this numerical example, the time window is chosen as one week and we have a severe “bear” market after the financial crisis of 2007-2008 during the selected period; hence the difference between investment returns $R(\omega) - r$ is small for most states $\omega \in \Omega$. With a longer time window and/or a better market performance, $R - r$ will increase, which causes opposite impact on Z_1 and Z_2 as λ increases; and hence we infer $K_1(Z_1)$ will be greater than 2.25 at certain model/market conditions when transaction costs are small.

On the other hand, despite $K_2(Z_2)$ is an increasing function of λ (then a decreasing function of $R - r$), $K_2(Z_2)$ is less sensitive to the change of λ or $R - r$ comparing to $K_1(Z_1)$. Therefore, in a “bull” market, we may have the case $\max\{K_1(Z_1), K_2(Z_2)\} = K_1(Z_1) > k$ for small λ , which corresponds to Case (3e) in Theorem 4.3, and then $\theta^* = \theta_M$. The economic interpretation for this scenario is that CPT investors should buy the risky asset as much as they can in a very good economy. If such scenario happens ($K_1(Z_1) > k$ for small λ), then the impact of transaction costs on the optimal investment θ^* is dramatic, because θ^* takes the upper bound θ_M when λ is less than a critical number λ_c , but directly plunges down to 0 when $\lambda > \lambda_c$.

6.3 The Case of $\alpha < \beta$

We next study the case of $\alpha < \beta$ when $\ln(1 + R)$ is normally distributed. In this case, the optimal investment is given by (3f) of Theorem 4.3. We investigate the impact of utility parameters, α and β , on optimal investment. In this particular study, we assume the investment constraint is not binding, and hence,

$$\theta^* = \arg \max_{\{\theta_1, \theta_2\}} J(\theta),$$

where θ_1 and θ_2 are given by 11 and 13, respectively. The proof of Theorem 4.3 provides conditions when $\theta^* = \theta_1$ or θ_2 .

First, we fix $\alpha = 0.88$, and calculate θ_1 and θ_2 as a function of β , where $0.88 < \beta < 1$. The transaction costs proportion λ is chosen at two levels, $\lambda = 0.2\%$ (representing small transaction costs) and $\lambda = 1\%$ (representing moderate to large transaction costs). From the results in Figure 4, we observe that θ_1 is an increasing function² of β , but θ_2 is a decreasing function of β . Furthermore, we obtain $\theta^* = \theta_1$ when $\lambda = 0.2\%$, but $\theta^* = \theta_2$ when $\lambda = 1\%$. This result shows that transaction costs have a big impact on the optimal investment, in fact, optimal strategies follow completely opposite directions (“buy” when transaction costs are small, “sell” when transaction costs are large). According to these two findings, the optimal investment (in absolute amount) increases as β increases (CPT investors become less risk averse towards losses).

In the next step, we fix $\beta = 0.88$ and consider $\alpha \in (0.6, 0.88)$. By following similar numerical calculations as in the previous study, we produce

²The increasing property of θ_1 with respect to β is not that noticeable in the graph of $\lambda = 1\%$, but is clearly supported by numerical results.

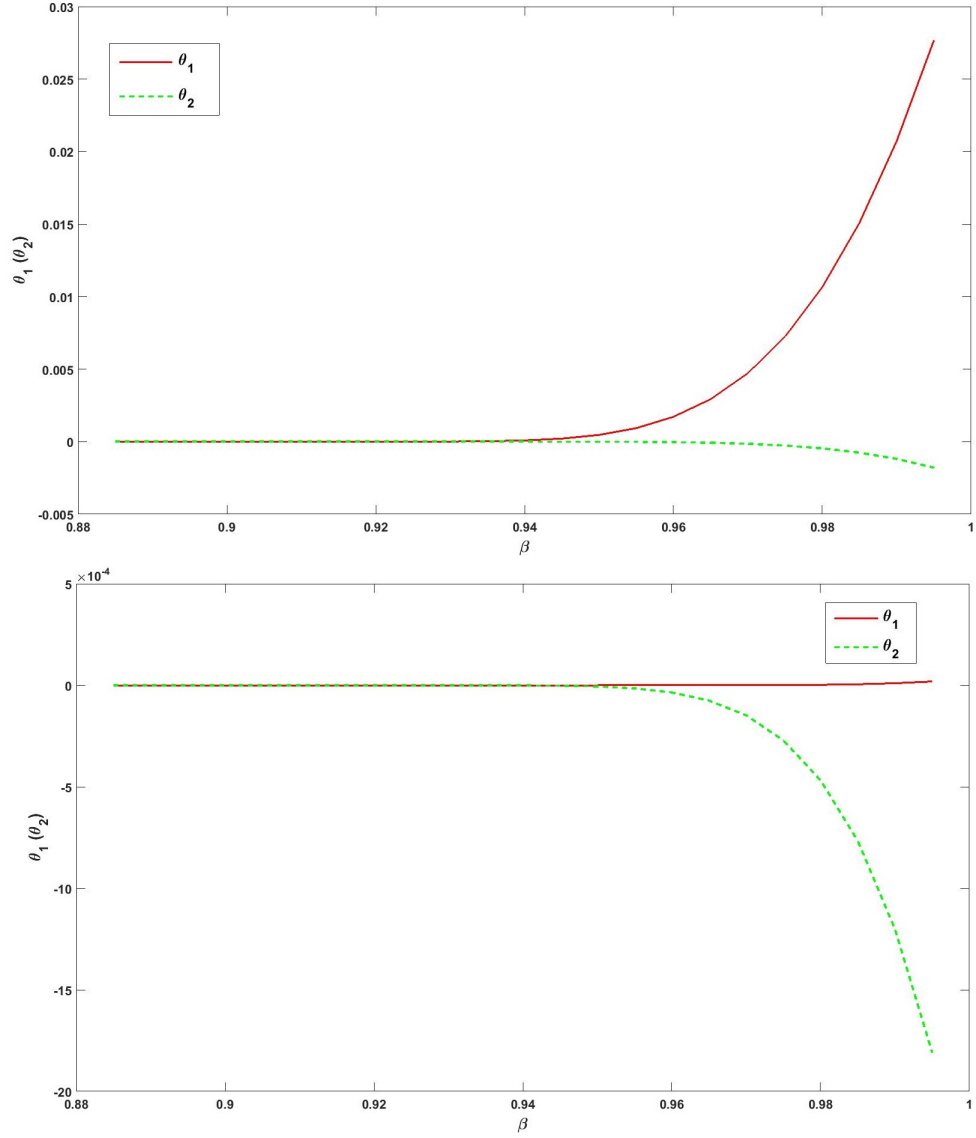


Figure 4: $\alpha = 0.88$, $0.88 < \beta < 1$, $\lambda = 0.2\%$ (above) and $\lambda = 1\%$ (below)

the results in Figure 5. Comparing with the findings from Figure 4, we obtain exactly opposite results regarding monotonicity. Namely, θ_1 is a decreasing function of α and θ_2 is an increasing function of α . As before, we still have $\theta^* = \theta_1$ under small transaction costs and $\theta^* = \theta_2$ under large transaction

costs. Therefore, this study shows that the optimal investment (in absolute amount) decreases as α increases (CPT investors become less risk averse towards gains).

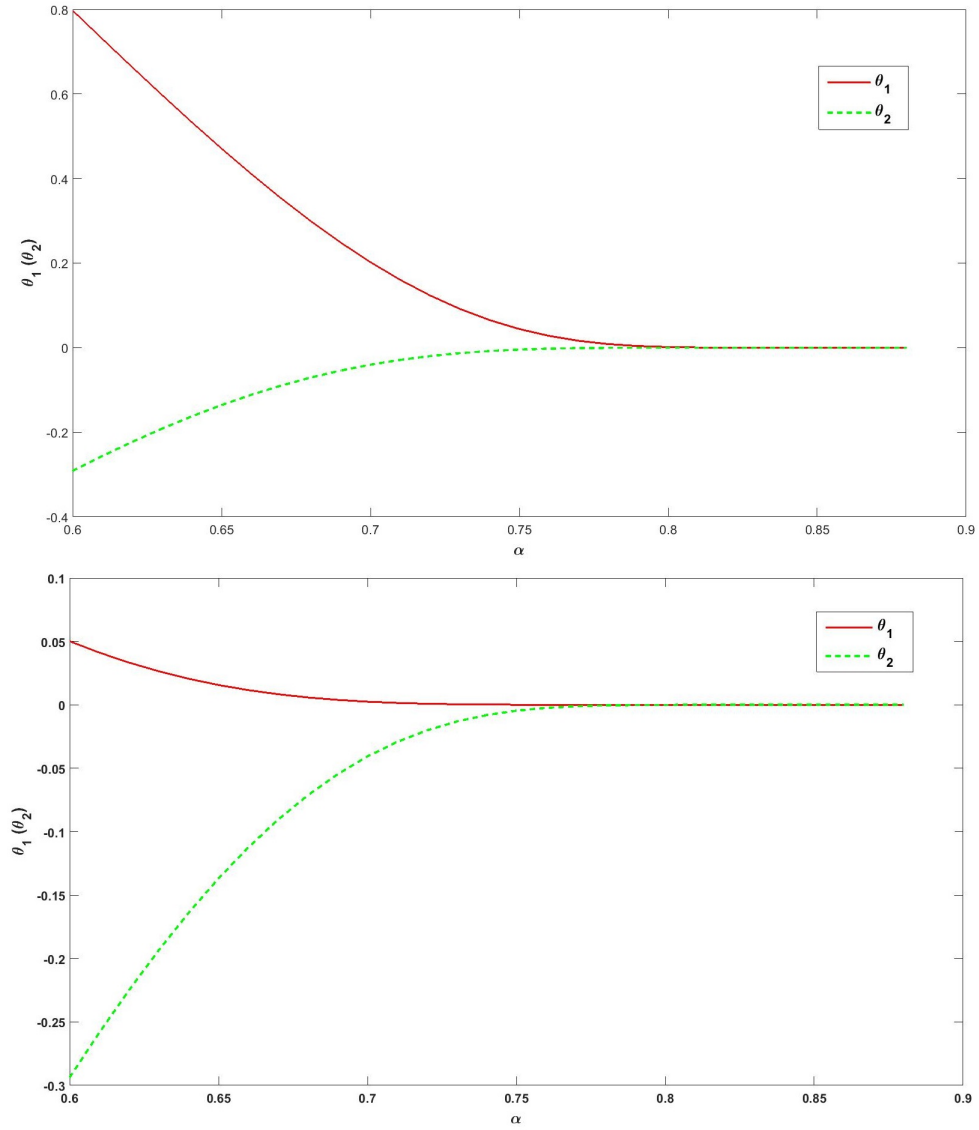


Figure 5: $0.6 < \alpha < 0.88$, $\beta = 0.88$, $\lambda = 0.2\%$ (above) and $\lambda = 1\%$ (below)

7 Conclusions

Prospect theory was proposed in Kahneman and Tversky (1979), and further developed as cumulative prospect theory (CPT) in Tversky and Kahneman (1992). The CPT states that, when facing uncertain outcomes, people make decisions based on the potential value of gains and losses rather than the final value as in the classical utility theory. In addition, people’s risk attitudes towards gains and losses are not universally risk averse, instead, they exhibit fourfold patterns (see Tversky and Kahneman (1992)):

risk aversion for gains and risk seeking for losses of high probability; risk seeking for gains and risk aversion for losses of low probability.

Those experimental findings challenge the fundamental axioms of expected utility theory, which, by far, is still the most popular criterion in economics and finance when it comes to decision making with uncertainty.

In this paper, we consider an investor who makes investment decisions according to CPT. We model the financial market by a single-period discrete time model, and take into account the transaction costs of trading the risky asset. The investor seeks optimal investment in the risky asset that maximizes the prospect value of his/her final random wealth.

The main objective of our work is to obtain explicit solution to the optimal investment problem with transaction costs under CPT. We have successfully obtained the optimal investment in explicit forms to this problem in two examples. We conduct economic analysis to study the impact of transaction costs and risk aversion on the optimal investment. The results confirm that transaction costs play an important role in the optimal investment. There exists threshold(s) of transaction costs proportion λ , in some cases the optimal investment is 0 when λ is above the threshold; while in other cases, the threshold separates the optimal investment into “buy” strategies and “sell” strategies. We also observe that the optimal investment is affected by investor’s risk aversion parameters. When investors become less risk averse towards losses, they will buy more or sell more risky asset. However, when investors become less risk averse towards gains, they will spend less amount in the risky asset.

References

- Barberis, N., and Huang, M., 2008. Stocks as lotteries: the implications of probability weighting for security prices. *American Economic Review* 98(5): 2066-2100.
- Bernard, C., and Ghossoub, M., 2010. Static portfolio choice under cumulative prospect theory. *Mathematics and Financial Economics* 2(4): 277-306.
- Bernoulli, D., 1954. Exposition of a new theory on the measurement of risk (translated by Louise Summer). *Econometrica* 22: 23-36.
- Carassus, L., and Rásonyi, M., 2015. On optimal investment for a behavioral investor in multiperiod incomplete market models. *Mathematical Finance* 25(1): 115-153.
- Carlier, G., and Dana, R., 2011. Optimal demand for contingent claims when agents have law invariant utilities. *Mathematical Finance* 21(2): 169-201.
- Davis, M., and Norman, A., 1990. Portfolio selection with transaction costs. *Mathematics of Operations Research* 15(4): 676-713.
- He, X.D., and Zhou, X.Y., 2011. Portfolio choice under cumulative prospect theory: an analytical treatment. *Management Science* 57(2): 315-331.
- Holt, A., and Laury, S., 2002. Risk aversion and incentive effects. *American Economic Review* 92(5): 1644-1655.
- Jin, H.Q., and Zhou, X.Y., 2008. Behavioral portfolio selection in continuous time. *Mathematical Finance* 18(3): 385-426.
- Kabanov, Y., and Safarian, M., 2010. *Markets with Transaction Costs*. Springer-Verlag Berlin Heidelberg.
- Kahneman, D., and Tversky, A., 1979. Prospect theory: an analysis of decision under risk. *Econometrica* 47(2): 263-291.
- Magill, M., and Constantinides, G., 1976. Portfolio selection with transaction costs. *Journal of Economic Theory* 13: 245-263.
- Merton, R., 1969. Lifetime portfolio selection under uncertainty: the continuous-time case. *The Review of Economics and Statistics* 51(3): 247-257.

- Pirvu, T., and Schulze, K., 2012. Multi-stock portfolio optimization under prospect theory. *Mathematics and Financial Economics* 6(4): 337-362.
- Prelec, D., 1998. The probability weighting function. *Econometrica* 66(3): 497-527.
- Quiggin, J., 1982. A theory of anticipated utility. *Journal of Economic Behavior and Organization* 3(4): 323-343.
- Rieger, M., and Bui, T., 2011. Too risk-averse for prospect theory? *Modern Economy* 2(4), 691-700.
- Rieger, M., and Wang, M., 2006. Cumulative prospect theory and the St. Petersburg paradox. *Economic Theory* 28(3): 665-679.
- Samuelson, P., 1969. Lifetime portfolio selection by dynamic stochastic programming. *The Review of Economics and Statistics* 51(3): 239-246.
- Shreve, S., and Soner, M., 1994. Optimal investment and consumption with transaction costs. *Annals of Applied Probability* 4(3): 609-692.
- Tversky, A., and Kahneman, D., 1992. Advances in prospect theory: cumulative representation of uncertainty. *Journal of Risk and Uncertainty* 5(4): 297-323.
- Von Neumann, J., and Morgenstern, O., 1944. *Theory of Games and Economic Behavior*. Princeton University Press.